Numerical integration methods (ODE+PDE)

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Numerical integration of differential equations

- We search a solution (a numerical approximation) for an ordinary differential equation

$$\frac{d^2y}{dt^2} = -ky$$

- Or a partial differential equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

- Given appropriate initial and boundary conditions

Numerical integration of ODEs

- NB: any ODE of order > 1 can be written as a system of first order equations Eg:

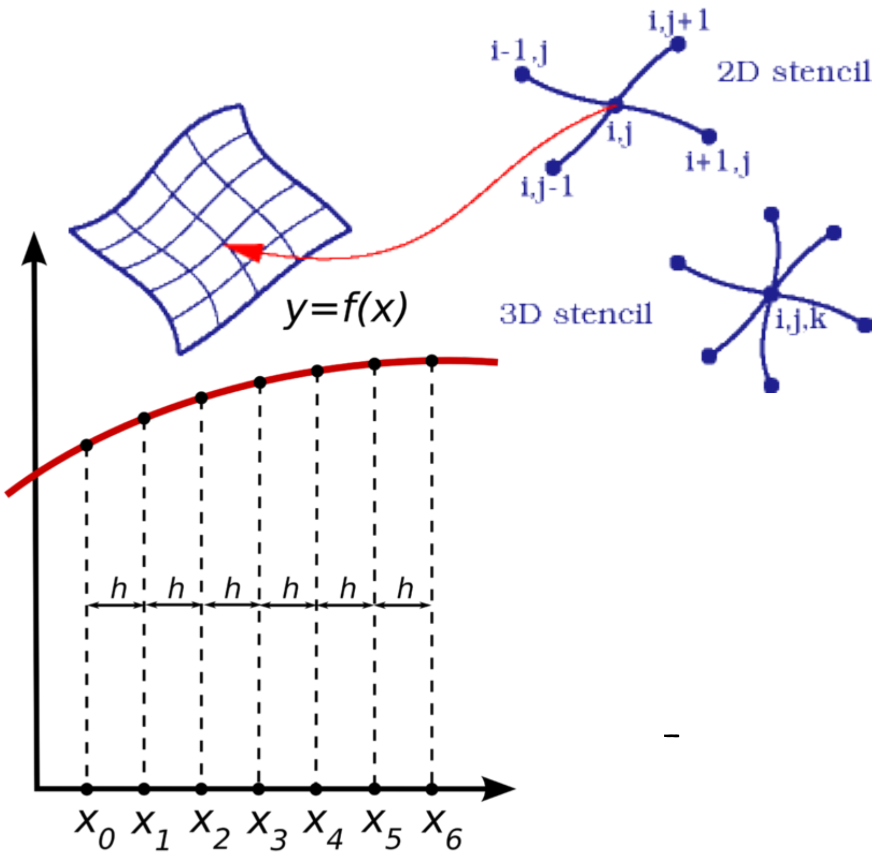
$$\frac{d^2y}{dt^2} = -ky \quad \Rightarrow \quad \begin{cases} \dfrac{du}{dt} = -ky \\ \dfrac{dy}{dt} = u \end{cases}$$

- General problem:

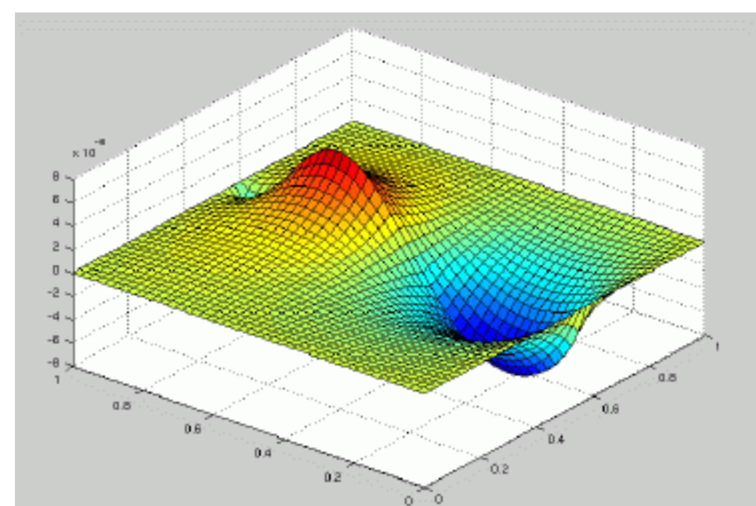
$$\frac{dy_i}{dt} = f_i(y_1, y_2 \ldots y_n; t)$$

Finite difference methods

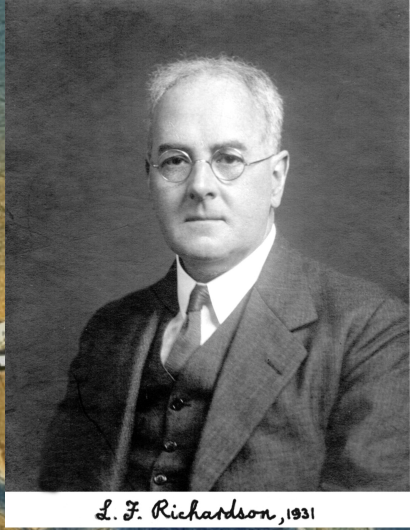
- We substitute to the continous problem a representation on a discretized grid (in space and time)



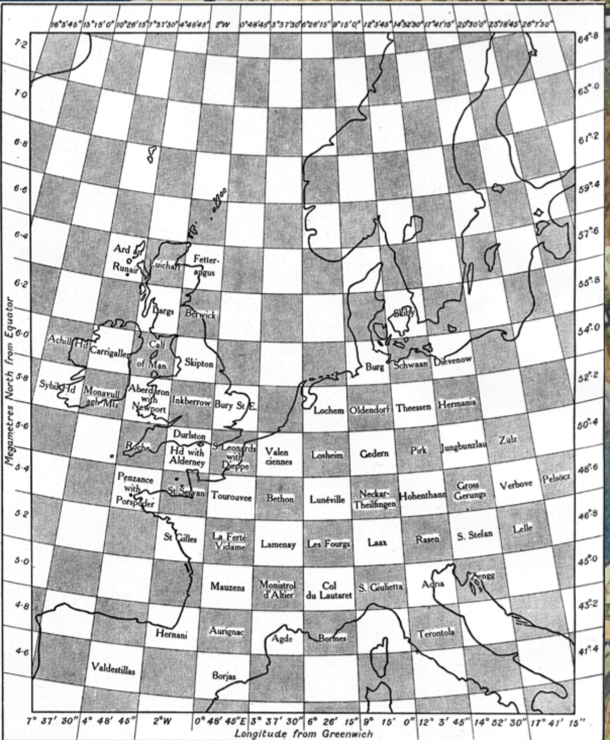
$$f'(a) \approx \frac{f(a+h) - f(a)}{h}.$$



L.F. Richardson, Weather Prediction by Numerical Process (1922):

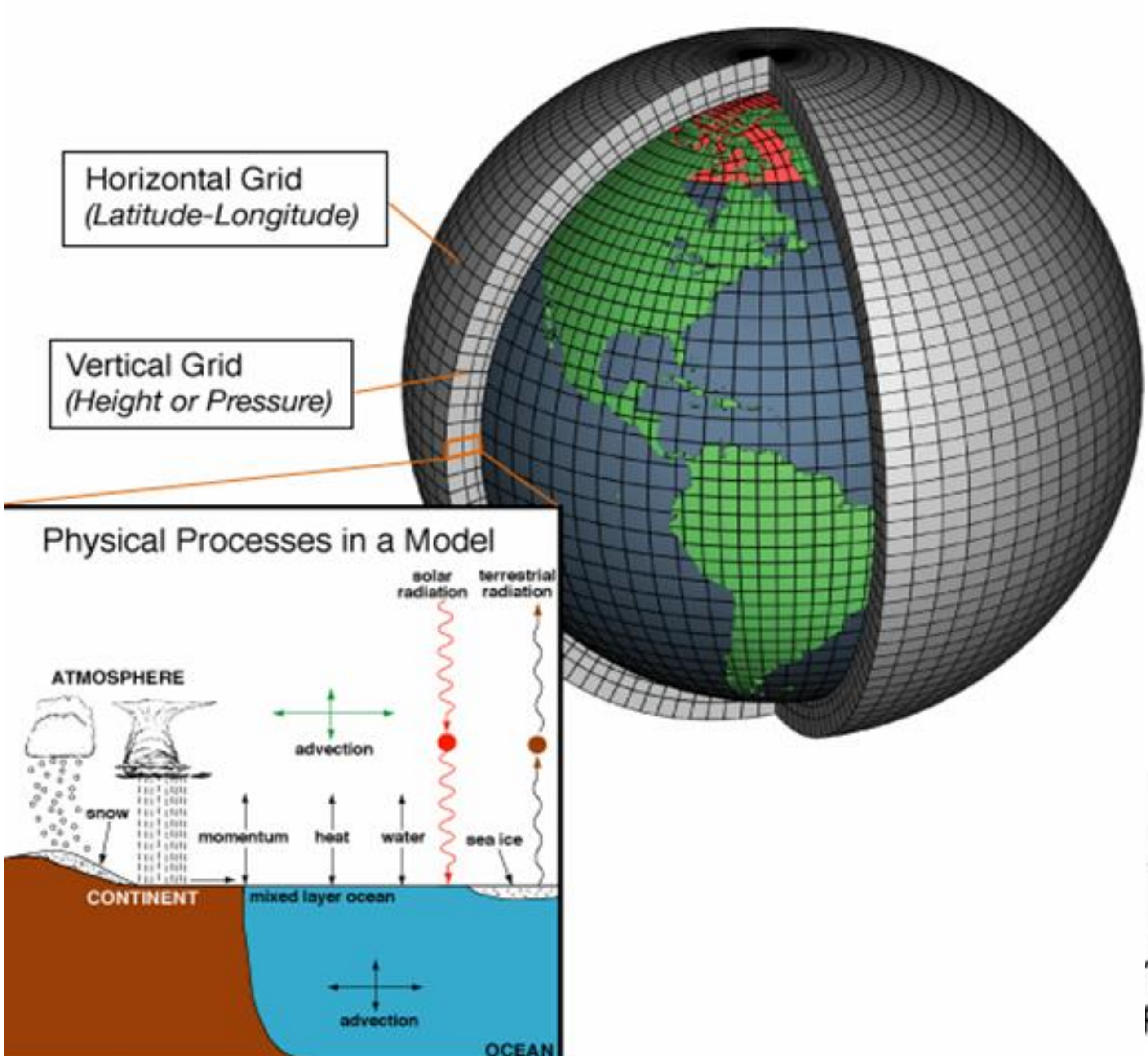


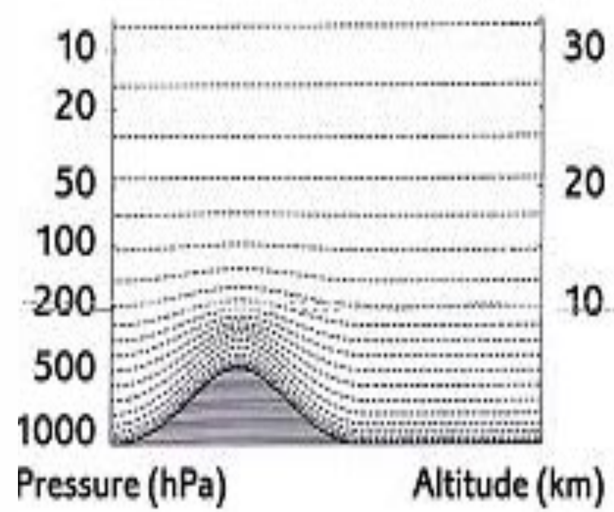
The forecast factory



"Imagine a large hall like a theater except that the circles and galleries go right round through the space usually occupied by the stage. The walls of this chamber are painted to form a map of the globe. . . . From the floor of the pit a tall pillar rises to half the height of the hall. It carries a large pulpit on its top. In this sits the man in charge of the whole theatre." (Weather Prediction by Numerical Process)

Numerical circulation models





Mathematical equations that represent the physical characteristics and processes are entered for each box

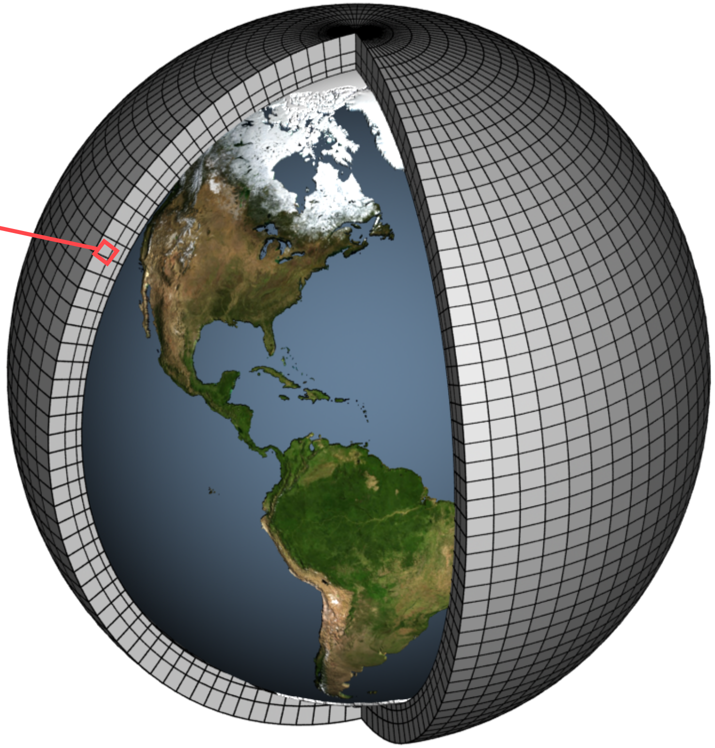
$$\frac{\partial u}{\partial t} = \eta v - \frac{\partial \Phi}{\partial x} - c_p \theta \frac{\partial \pi}{\partial x} - z \frac{\partial u}{\partial \sigma} - \frac{\partial (\frac{u^2+v^2}{2})}{\partial x}$$

$$\frac{\partial v}{\partial t} = -\eta \frac{u}{v} - \frac{\partial \Phi}{\partial y} - c_p \theta \frac{\partial \pi}{\partial y} - z \frac{\partial v}{\partial \sigma} - \frac{\partial (\frac{u^2+v^2}{2})}{\partial y}$$

$$\frac{\delta T}{\partial t} = \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}$$

$$\frac{\delta W}{\partial t} = u \frac{\partial W}{\partial x} + v \frac{\partial W}{\partial y} + w \frac{\partial W}{\partial z}$$

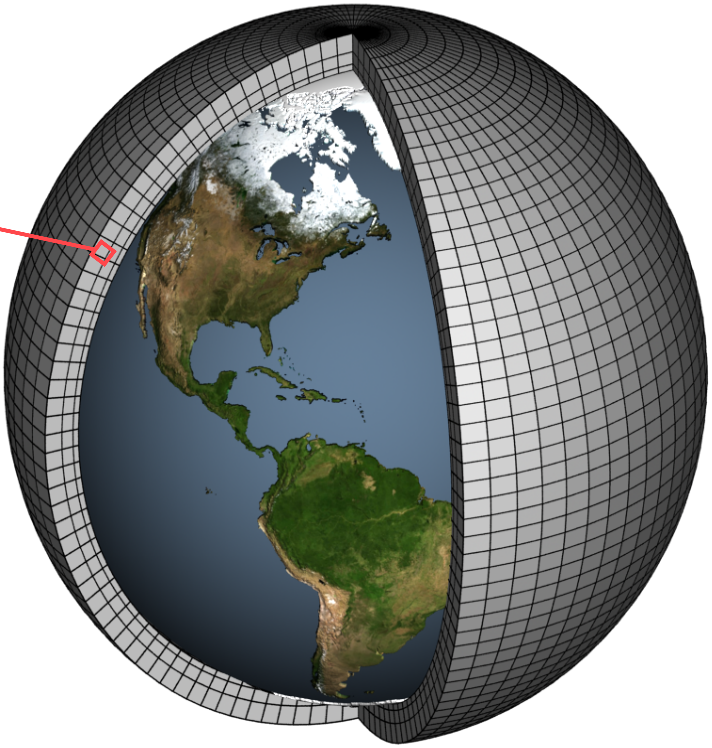
$$\frac{\partial}{\partial t} \frac{\partial p}{\partial \sigma} = u \frac{\partial}{\partial x} x \frac{\partial p}{\partial \sigma} + v \frac{\partial}{\partial y} y \frac{\partial p}{\partial \sigma} + w \frac{\partial}{\partial z} z \frac{\partial p}{\partial \sigma}$$



(da una lezione di S. Gualdi – CMCC)

Equations are converted to computer code and climate variables are set

```
if (diagts .and. eots) then
  do 1500 m=1,nt
    do 1490 k=1,km
      fx = cst(j)*dyt(j)*dzt(k)/(c2dtts*dtxcel(k))
      do 1480 i=2,imtm1
        boxfx             = fx*dxt(i)*fm(i,k,jc)
        sddt              = (ta(i,k,m)-t(i,k,jc,nm,m))*boxfx
        svar              = (ta(i,k,m)**2-t(i,k,jc,nm,m)**2)
                            *boxfx
        n                 = 0
        termbt(k,1,m,n) = termbt(k,1,m,n) + sddt
        tvar(k,m,n)      = tvar(k,m,n)      + svar
        n       = nhreg*(mskvr(k)-1) + mskhr(i,j)
        if (n .gt. 0 .and. mskhr(i,j) .gt. 0) then
          termbt(k,1,m,n) = termbt(k,1,m,n) + sddt
          tvar(k,m,n)      = tvar(k,m,n)      + svar
```



(da una lezione di S. Gualdi – CMCC)

Finite differences representation of derivatives

Can be obtained from

- Classical definition of first order derivative of a function u(x,y) at a point:

$$\frac{\partial u}{\partial x}|_o = lim_{\Delta x \to 0} \frac{u(x_o + \Delta x, y_o) - u(x_o, y_o)}{\Delta x}$$

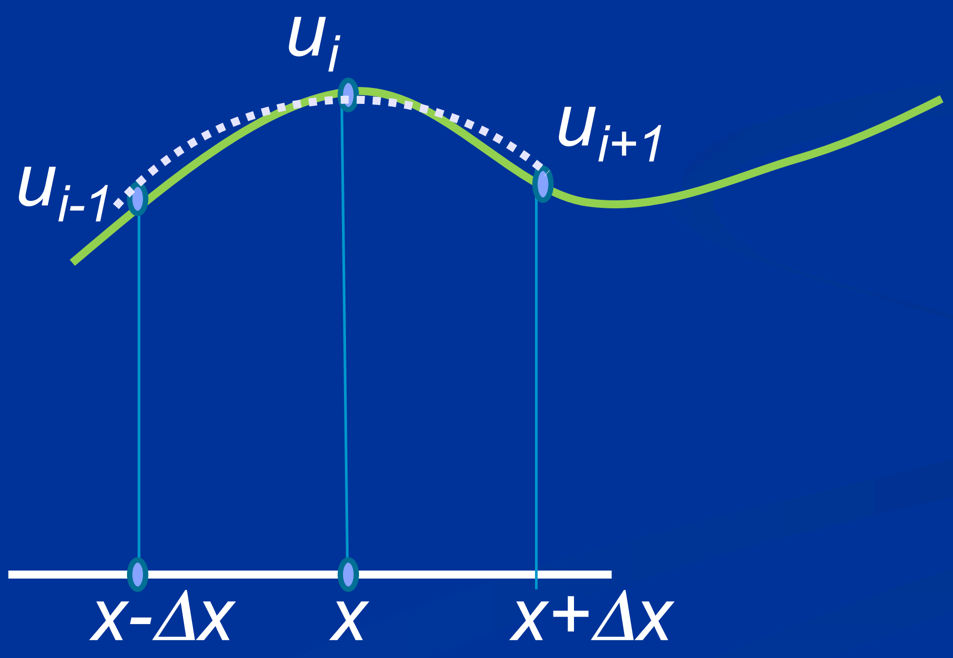
- Taylor expansion of u(x,y) around a point

$$u(x_o + \Delta x, y_o) = u(x_o, y_o) + \Delta x \frac{\partial u}{\partial x}|_o + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2}|_o + O(\Delta x^3)\ldots$$

Finite differences representation of derivatives

Can be obtained from:

- Fitting a polynomial around a point::



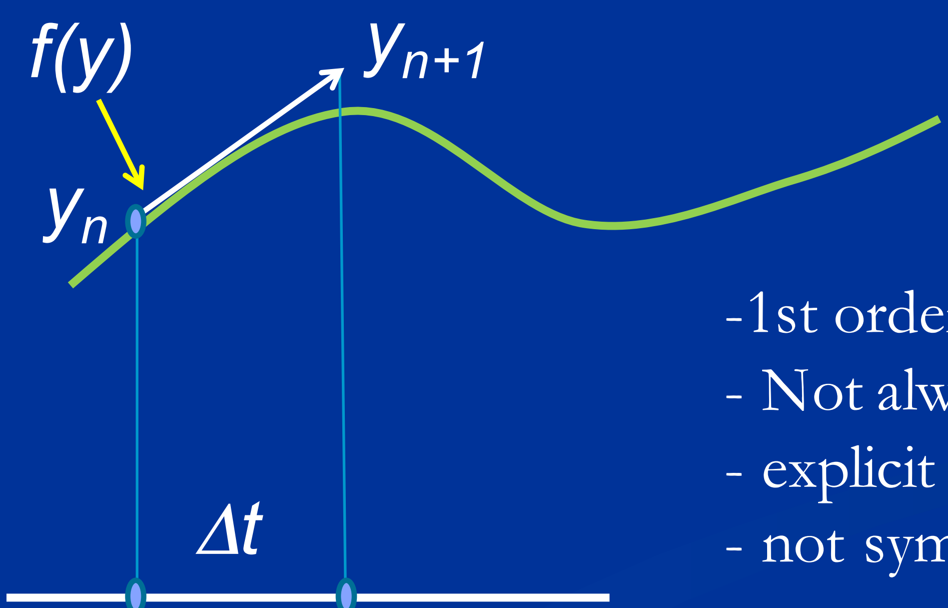
ODE: The Euler method

$$\frac{dy}{dt} = f(y)$$

$$y(t + \Delta t) = y(t) + \Delta t\, f(y(t))$$

$$y_{n+1} = y_n + \Delta t\, f(y_n)$$

+ Err.ore Troncamento



-1st order method accura
- Not always stable
- explicit
- not symmetric

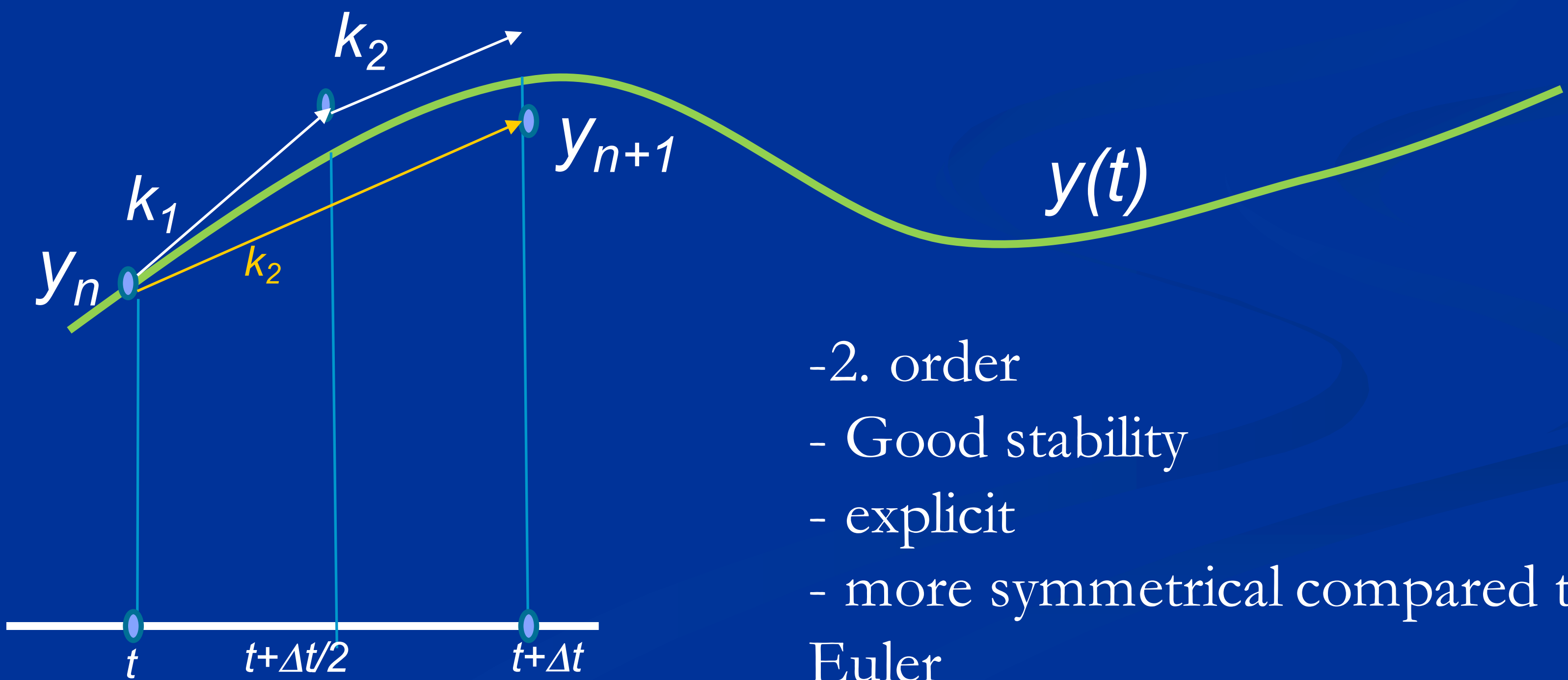
ODE: Runge-Kutta 2

C.D.T. Runge

M. W. Kutta

$$\frac{dy}{dt} = f(y)$$

$$\begin{cases} k_1 = f(y) \\ k_2 = f\left(y + \frac{\Delta t}{2} k_1\right) \\ y(t + \Delta t) = y(t) + k_2 \Delta t + O(\Delta t^3) \end{cases}$$



-2. order
- Good stability
- explicit
- more symmetrical compared to Euler

ODE: Runge-Kutta 4

$$\frac{dy}{dt} = f(y)$$

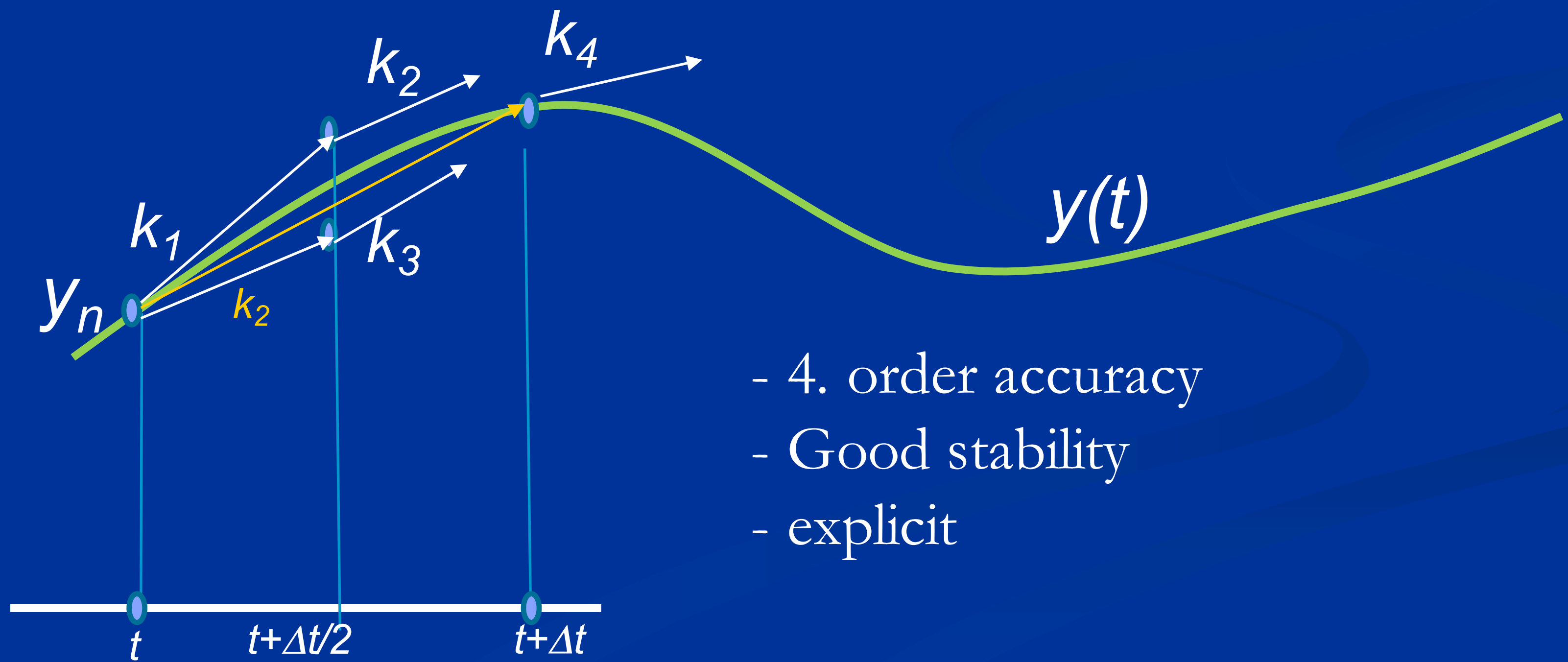
$$k_1 = f(y)$$

$$k_2 = f\left(y + \frac{\Delta t}{2} k_1\right)$$

$$k_3 = f\left(y + \frac{\Delta t}{2} k_2\right)$$

$$k_4 = f\left(y + \Delta t k_3\right)$$

$$y(t+\Delta t) = y(t) + \left[\frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}\right] \Delta t + O(\Delta t^5)$$



- 4. order accuracy
- Good stability
- explicit

Other methods

- Leapfrog
- …. Many other explicit methods
- Predictor-corrector
- Implicit methods (greater stability, non necessarily more accurate)

PDE, examples

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$$

Scalar advection

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -v\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

FT CS

Non è stabile!

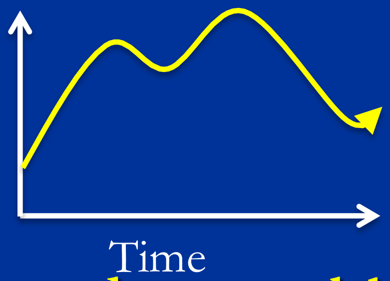
$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}$$

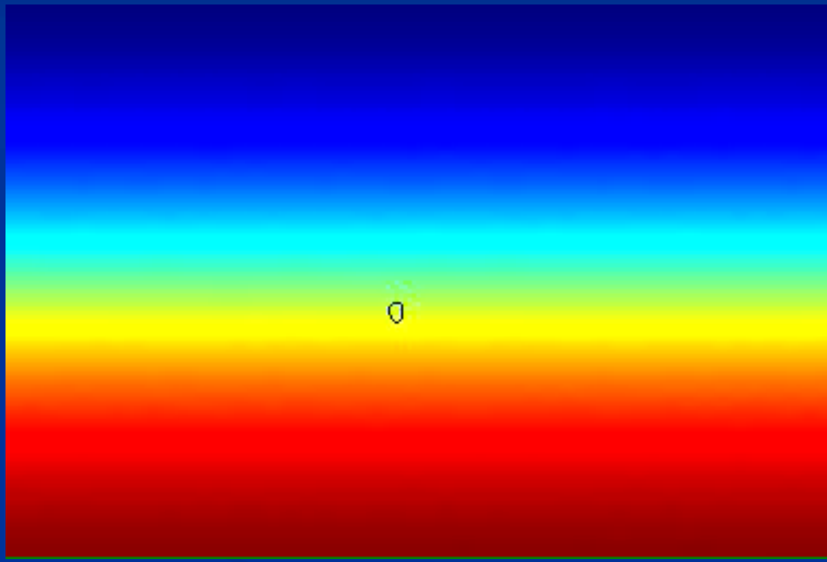
Heat equation

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \nu\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

Initial condition problems vs. boundary problems

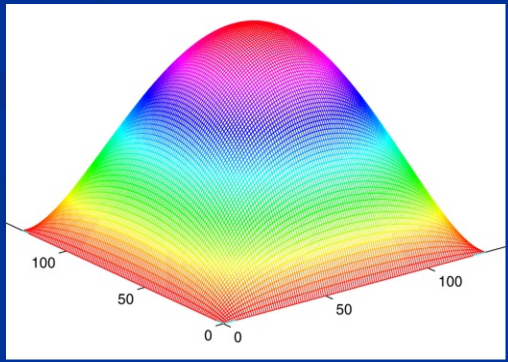
- Initial conditions problems:
  - Choice of initial conditions
  - Evolution equations
  - Boundary conditions





- Boundary problems
  - Equations to be solved in the domain
  - Boundary conditions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho$$



PDE: Boundary conditions

- Dirichlet conditions. $u=f$ su $\partial\Omega$
- von Neumann conditions:
  eg: $\partial u/\partial n=f$ or $\partial u/\partial s=g$ in $\partial\Omega$
- Mixed conditions e.g: $\partial u/\partial n+ku=f$